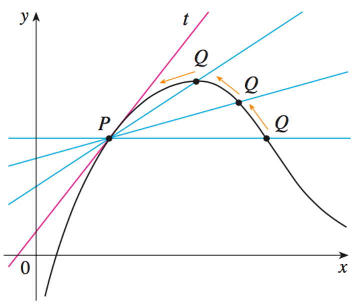
LECTURE: 2-7 DERIVATIVES AND RATES OF CHANGE

Tangents

> The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope
> $$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
> provided this limit exists.



- Let Q be a point $Q(x, f(x))$
- The slope of the secant line is $\frac{f(x) - f(a)}{x - a}$
- The slope of the tangent line is $\lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a}$

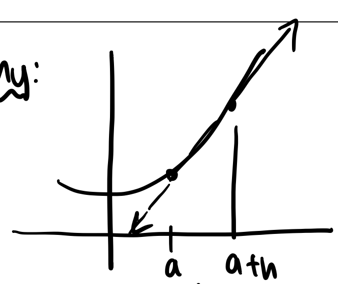
Example 1: Find an equation of the tangent line to $y = x^2$ at the point $(2, 4)$.

$$m = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$
$$= \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$
$$= \lim_{x \to 2} \frac{(x+2)(x-2)}{x - 2}$$
$$= \lim_{x \to 2} (x+2)$$
$$= \boxed{4}$$

equation: $y - y_1 = m(x - x_1)$
$$y - 4 = 4(x - 2)$$
$$y - 4 = 4x - 8$$
$$\boxed{y = 4x - 4}$$

> **An Alternative Expression for the Slope of the Tangent Line:**
> $$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

why:



as $h \to 0$ $a+h \to a$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{a + h - a}$$
$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 2: Find an equation of the tangent line to $y = 2/x$ at the point $(1, 2)$.

$$m = \lim_{x\to 1} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x\to 1} \frac{2/x - 2/1}{x-1}$$

$$= \lim_{x\to 1} \frac{2-2x}{x} \cdot \frac{1}{x-1}$$

$$= \lim_{x\to 1} \frac{2(1-x)}{x} \cdot \frac{1}{(x-1)}$$

$$= \lim_{x\to 1} \frac{-2}{x}$$

$\boxed{m = -2}$ ← slope

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -2(x-1)$$

$$y - 2 = -2x + 2$$

$$\boxed{y = -2x + 4}$$

Velocities

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. How would you find the instantaneous velocity $v(a)$ at time $t = a$?

$$v(a) = \lim_{t\to a} \frac{f(t) - f(a)}{t-a} \quad \text{or} \quad \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Example 3: If a ball is thrown into the air with a velocity if 40 ft/sec, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = a$ and use this to find the velcoity at $t = 1$ and $t = 2$.

$$v(a) = \lim_{h\to 0} \frac{40(a+h) - 16(a+h)^2 - (40a - 16a^2)}{h}$$

$$= \lim_{h\to 0} \frac{40a + 40h - 16(a^2 + 2ah + h^2) - 40a + 16a^2}{h}$$

$$= \lim_{h\to 0} \frac{40h - 32ah - 16h^2}{h}$$

$$= \lim_{h\to 0} (40 - 32a - 16h)$$

$$= \boxed{40 - 32a}$$

$$v(1) = 40 - 32 = \boxed{8 \text{ ft/sec}}$$

$$v(2) = 40 - 64 = \boxed{-24 \text{ ft/sec}}$$

Derivatives

The **derivative of a function** f **at a number** a, denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

Example 4: Find the derivative of $f(x) = 5 - 2x - x^2$. Then, find an equation of the tangent line to $f(x)$ at the point $(1, 2)$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{5 - 2(a+h) - (a+h)^2 - (5 - 2a - a^2)}{h}$$

$$= \lim_{h\to 0} \frac{5 - 2a - 2h - (a^2 + 2ah + h^2) - 5 + 2a - a^2}{h}$$

$$= \lim_{h\to 0} \frac{-2h - 2ah - h^2}{h}$$

$$= \lim_{h\to 0} (-2 - 2a - h)$$

$$= \boxed{-2 - 2a}$$

at $(1,2)$

$m = f'(1)$

$m = -2 - 2$

$m = -4$

$y - y_1 = m(x - x_1)$

$y - 2 = -4(x - 1)$

$y - 2 = -4x + 4$

$\boxed{y = -4x + 6}$

Example 5: Given $f(x) = x^2 + \dfrac{2}{x}$ find $f'(a)$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \left[(a+h)^2 + \frac{2}{a+h} - \left(a^2 + \frac{2}{a}\right)\right]\frac{1}{h}$$

$$= \lim_{h\to 0} \left[a^2 + 2ah + h^2 - a^2 + \frac{2}{a+h} - \frac{2}{a}\right]\frac{1}{h}$$

$$= \lim_{h\to 0} \left[2ah + h^2 + \frac{2}{a(a+h)} - \frac{2(a+h)}{a(a+h)}\right]\frac{1}{h}$$

$$= \lim_{h\to 0} \left[2ah + h^2 + \frac{2a - 2a - 2h}{a(a+h)}\right]\frac{1}{h}$$

$$= \lim_{h\to 0} \left(2a + h - \frac{2}{a(a+h)}\right)$$

$$= \boxed{2a - \frac{2}{a^2}}$$

Example 6: The displacement (in feet) of a particle moving in a straight line is given by $s(t) = \frac{1}{2}t^2 - 6t + 23$, where t is measured in seconds.

(a) Find the average velocity over each time interval.

(i) $[4, 8]$

$$\frac{s(8)-s(4)}{8-4} = \frac{\frac{1}{2}(64) - 48 + 23 - (\frac{1}{2}(16) - 24 + 23)}{4}$$

$$= \frac{+32 - 48 + \cancel{23} - (+8 - 24 + \cancel{23})}{4}$$

$$= \boxed{0 \text{ ft/sec}}$$

(ii) $[6, 8]$

$$\frac{s(8)-s(6)}{8-6} = \frac{32 - 48 + 23 - (18 - 36 + 23)}{2}$$

$$= \frac{-16 - 18 + 36}{2}$$

$$= \frac{-2}{2} = \boxed{-1 \text{ ft/sec}}$$

(b) Find the instantaneous velocity when $t = 8$.

$$v(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2}(a+h)^2 - 6(a+h) + 23 - \frac{1}{2}a^2 + 6a - 23}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2}(a^2 + 2ah + h^2) - 6a - 6h - \frac{1}{2}a^2 + 6a}{h}$$

$$= \lim_{h \to 0} \frac{ah + \frac{1}{2}h^2 - 6h}{h}$$

$$= \lim_{h \to 0} (a + \tfrac{1}{2}h - 6) = \boxed{a - 6}$$

at $t = 8$

$$v(8) = 8 - 6$$

$$= \boxed{2 \text{ ft/sec}}$$

Rates of Change

Example 7: The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars.

(a) What is the meaning of the derivative $f'(x)$? What are its units?

- $f'(x)$ is the rate of change of the production with respect to the number of ounces of gold produced.
- It's units are dollars per ounce.

(b) What does the statement $f'(800) = 17$ mean?

- when 800 ounces of gold have been produced, production costs are increasing at \$17 per ounce.
- Producing the 800th (or 801st oz) will cost about \$17.

(c) Do you think that the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

- In the short term the values of $f'(x)$ will decrease because the start up costs are spread out.
- In the long term costs may increase as you scale up the operation or find all the easy gold.

Example 8: The table below shows world average daily oil consumption from 1985 to 2010 measured in thousands of barrels per day.

(a) Compute and interpret the average rate of change from 1990 to 2005. What are the units?

Years since 1985	Thousands of barrels of oil per day
0	60,083
5 1990	66,533
10	70,099
2000 ← 15	76,784
20 2005	84,077
25	87,302

$$\frac{84077-66533}{2005-1990} = \boxed{1169.6 \text{ thousands of barrels per day per year}}$$

The rate of change of oil production is increasing at 1169.6 thousands of barrels per day each year.

(b) Estimate the instantaneous rate of change in 2000 by taking the average of two average rates of change. What are its units?

use 1995 + 2000 $\rightarrow \dfrac{76784-70099}{15-10} = 1337$

use 2000 + 2005 $\rightarrow \dfrac{84077-76784}{20-15} = 1458.6$

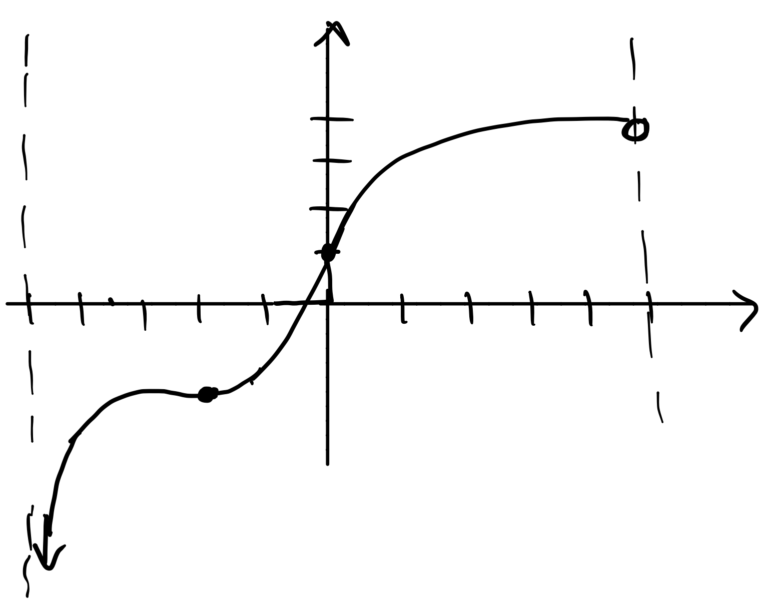
avg: $\dfrac{1337+1458.6}{2} = \boxed{1397.8 \text{ thousands of barrels per day per year}}$

Example 9: If an equation of the tangent line to the curve $y = f(x)$ at the point where $a = 1$ is $y = -7x + 2$, find $f(1)$ and $f'(1)$.

$f'(1)$ is the slope of the tangent line, so $f'(1) = -7$

$f(1) = -7 + 2 = \boxed{-5}$ because the tangent line intersects the curve at the point of tangency.

Example 10: Sketch the graph of a function f which is continuous on the domain $(-5, 5)$ and where $f(0) = 1$, $f'(0) = 1$, $f'(-2) = 0$, $\lim_{x\to-5^+} f(x) = -\infty$, and $\lim_{x\to5^-} f(x) = 4$



- $f(0)=1 \rightarrow f$ passes through $(0,1)$
- $f'(0)=1 \rightarrow f$ has slope 1 @ $x=0$
- $f'(-2)=0 \rightarrow f$ has slope 0 @ $x=-2$
- $\lim_{x\to-5^+} f(x) = -\infty$ means $f(x)\rightarrow-\infty$ as $x\rightarrow-5^+$